Hecke operators For congruence subgroups $\Gamma_0(q)$ and χ mod q 1.

Def For $k \in \mathbb{Z}$, $A \in GL_2^+(\mathbb{R})$ positive determinant

we define the slash operator on functions $f: \mathcal{H} \to \mathbb{C}$ by

$$f|_A(z) = (\det A)^{k/2} j_A(z)^{-k} f(Az)$$

where we recall that $j_A(z) = (cz+d)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Exercise

$$j_{AB}(z) = j_A(Bz) j_B(z)$$

$$f|_{AB}(z) = (f|_A)|_B$$

$$f|_{aA} = \left(\frac{a}{|a|}\right)^k f|_A \qquad (a \in \mathbb{R})$$

Hecke operators T_n $\Gamma = SL_2(\mathbb{Z})$ modular group

For $n \in \mathbb{Z}_{\geq 1}$, define the set

$$G_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = n \right\}$$

$G_1 = \Gamma = SL_2(\mathbb{Z})$

Γ acts on G_n on both sides, and we can look at the right or left cosets.

lemma $\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n, \boxed{0 \leq b < d} \right\}$

is a complete set of right coset representatives of G_n modulo Γ ie $G_n = \dot{\bigcup}_{\rho \in \Delta_n} \Gamma\rho$

Remark

$|\Delta_n| = \sum_{d|n} d$

Only positive divisors because of $\boxed{}$

Proof $g = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in G_n$. $\exists \gamma, \delta$ st $\gamma a + \delta c = 0$, and $(\gamma, \delta) = 1$

ie $\exists\ \tau = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$ st $\tau g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ wlog $ad = n$ with $d > 0$

Then $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \tau g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b+ud \\ 0 & d \end{pmatrix}$ $0 \leq b' < d$ Division Algo Divide b by d

ie $Ag = g_n$, $g_n \in \Delta_n \Rightarrow g = A^{-1} g_n \in \Gamma g_n$.

The cosets Γg_n are disjoint since

$$\underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in \Gamma} \underbrace{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}_{\in \Delta_n} = \underbrace{\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}}_{\in \Delta_n} \Rightarrow \gamma = 0, \ \alpha = \delta = 1$$

and $\beta = 0$ (since $0 \leq b' < d'$)

Remark This would work similarly for left cosets, and for any $\rho \in \Delta_n$, $\tau \in \Gamma$, $\exists\ \rho', \tau'$ st $\rho\tau = \tau'\rho'$ ($\exists$ unique)

Also, for a fixed τ, as ρ range over Δ_n, so does ρ'.

Writing
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} \alpha & * \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$$
$$\iff \begin{pmatrix} \alpha a + \gamma b & * \\ d\gamma & d\delta \end{pmatrix} = \begin{pmatrix} \alpha' a' & * \\ \gamma' a' & \gamma' b' + \delta' c' \end{pmatrix}$$

Since $(\alpha', \gamma') = 1$ (determinant is 1), we get

$$a' = (\alpha' a', \gamma' a') = (\alpha a + \gamma b, \gamma d)$$
$$\Rightarrow \alpha' = \frac{\alpha a + \gamma b}{(\alpha a + \gamma b, \gamma d)} \quad \& \quad \gamma' = \frac{\gamma d}{(\alpha a + \gamma b, \gamma d)}$$

a', d', α', γ' are uniquely determined.

and $a'd' = n \Rightarrow d' = \dfrac{n}{(\alpha a + \gamma b, \gamma d)}$

Now we have to show that $\boxed{\delta' \text{ and } d'}$ exist and are unique.

Since $\det\begin{pmatrix} \alpha' & * \\ \gamma' & \delta' \end{pmatrix} = 1 \Rightarrow \alpha'\delta' \equiv 1 \mod |\gamma'|$
$$\Rightarrow \delta' \equiv (\alpha')^{-1} \mod |\gamma'|$$

Since $\delta d = \gamma' b' + \delta' d'$

$\Rightarrow \gamma' b' \equiv \delta d \mod d' \Rightarrow b' \equiv \gamma'^{-1} \delta d \mod d'$

and then if $0 \leq b' < d'$, then b' is unigely determined and then also δ' since
$$\delta' d' = \delta d - \gamma' b'$$

Let $\chi : \mathbb{Z} \to \mathbb{C}$ be any function (it will be a Dirichlet character at some point).

Def For $g \in GL_2(\mathbb{Z})$, $\chi(g) = \overline{\chi}(a)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Def The operator $T_n : \{f : \mathcal{H} \to \mathbb{C}\} \longrightarrow \{f : \mathcal{H} \to \mathbb{C}\}$ is given by

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho) f|_\rho$$

$$= n^{\frac{k}{2}-1} \sum_{ad=n} \chi(a) n^{k/2} d^{-k} \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right)$$

notation has to do with cusps.

$$= \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right)$$

Let $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{Z} \right\}$, and assume $\chi(1) = 1$ & $\chi(-1) = (-1)^k$

and $M_k(\Gamma_\infty, \chi) = \{ f : \mathcal{H} \to \mathbb{C} : f|_\tau = \chi(\tau) f \quad \forall \tau \in \Gamma_\infty \}$

$$= \{ f : \mathcal{H} \to \mathbb{C} : f(z+1) = f(z) \}$$

Thm $T_n : M_k(\Gamma_\infty, \chi) \longrightarrow M_k(\Gamma_\infty, \chi)$

proof $\forall \rho \in \Delta_n$ & $\tau \in \Gamma_\infty$, $\exists \rho', \tau'$ st $\rho\tau = \tau'\rho'$

ie $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$ $\quad (b' = b + au - dv)$

Then $f|_{\rho\tau} = f|_{\tau'\rho'} = (f|_{\tau'})_{\rho'} = (\det(\tau')^{k/2} j_{\tau'}(z)^{-k} f(\tau' z))|_{\rho'}$

$$= \chi(\tau') f|_{\rho'}$$

Then

$$T_n(f)|_\tau = \left[n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho) f|_\rho \right]\Big|_\tau$$

$$= n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho) f|_{\rho\tau} = n^{\frac{k}{2}-1} \sum_{\rho' \in \Delta_n} \overline{\chi}(\rho) f|_{\tau'\rho'}$$

$\chi(\rho)\chi(\tau)$
$= \chi(\rho')\chi(\tau')$
$\Rightarrow \overline{\chi}(\rho)\chi(\tau')$
$= \overline{\chi}(\rho')\chi(\tau)$
(mult by conjugate and commute)

$$= n^{\frac{k}{2}-1} \sum_{\rho' \in \Delta_n} \overline{\chi}(\rho) \chi(\tau') f|_{\rho'}$$

$$= n^{\frac{k}{2}-1} \sum_{\rho' \in \Delta_n} \overline{\chi}(\rho') \chi(\tau) f|_{\rho'} = \chi(\tau) T_n(f).$$

The action of the Hecke operator on $f(z) = \sum_{m=0}^{\infty} a(m) e(mz)$

<u>Prop</u> $$(T_n f)(z) = \sum_{m=0}^{\infty} a_n(m)\, e(mz)$$

where $$a_n(m) = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, a\left(\frac{mn}{d}\right)$$

<u>Remark</u> If $(m,n)=1$, $a_n(m) = a(mn)$. Also $a_n(m) = a_m(n)$.

<u>proof</u> $$(T_n f)(z) = \frac{1}{n} \sum_{m=0}^{\infty} a(m) \sum_{ad=n} \chi(a)\, a^k \sum_{\substack{0 \le b < d \\ b \bmod d}} e\left(m \frac{az+b}{d}\right)$$

$$= \frac{1}{n} \sum_{m=0}^{\infty} a(m) \sum_{ad=n} \chi(a)\, a^k\, e\left(\frac{maz}{d}\right) \boxed{\sum_{\substack{0 \le b < d \\ b \bmod d}} e\left(\frac{mb}{d}\right)} = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$$

Exercise:

$\boxed{m = \ell d}$

$$= \frac{1}{n} \sum_{\ell=0}^{\infty} \sum_{ad=n} a(\ell d)\, \chi(a)\, a^k\, d\, e(\ell a z) \qquad d \to \frac{n}{a}$$

$$= \sum_{\ell=0}^{\infty} \sum_{ad=n} a(\ell d)\, \chi(a)\, a^{k-1}\, e(a \ell z)$$

$$= \sum_{m=0}^{\infty} \left[\sum_{\substack{ad=n \\ a\ell=m}} a(\ell d)\, \chi(a)\, a^{k-1} \right] e(mz)$$

$$= \sum_{m=0}^{\infty} \left[\sum_{a \mid (m,n)} \chi(a)\, a^{k-1}\, a\left(\frac{mn}{a^2}\right) \right] e(mz)$$

<u>Cor</u> For $n \ge 1$, $$a_n(0) = \sum_{d \mid n} \chi(d)\, d^{k-1}\, a(0) = \sigma_{k-1}(n, \chi)\, a(0)$$

We now assume that χ is completely multiplicative

<u>ie</u> $\chi(a_1 a_2) = \chi(a_1)\chi(a_2)$

<u>Thm</u> For any $m, n \ge 1$, $$T_m T_n = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}$$

<u>proof</u> By def of T_m,

$$mn\, T_n T_m = \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} \chi(a_1 a_2)(a_1 a_2)^k \sum_{\substack{b_1 (d_1) \\ b_2 (d_2)}} \underbrace{\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}}_{= \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{pmatrix}}$$

[just the operator]

Use $\delta = (a_1, d_2)$

$$a_1' = \frac{a_1}{\delta} \qquad d_2' = \frac{d_2}{\delta} \qquad (a_1', d_2') = 1$$

$$= \sum_{\substack{\delta \mid mn \\ (a_1', d_2') = 1}} \chi(\delta a_1' a_2) \sum_{\substack{a_1' d_1 = m/\delta \\ a_2 d_2' = m/\delta}} (\delta a_1' a_2)^k \sum_{\substack{b_1 (d_1) \\ b_2 (\delta d_2')}} \begin{pmatrix} \delta a_1' a_2 & b \\ 0 & \delta d_1 d_2' \end{pmatrix}$$

$$= \sum_{\substack{\delta \mid mn \\ (a_1, d_2) = 1}} \chi(\delta) \delta^k \sum_{\substack{a_1 d_1 = m/\delta \\ a_2 d_2 = m/\delta}} (a_1 a_2)^k \sum_{\substack{b_1 (d_1) \\ b_2 (\delta d_2)}} \delta \begin{pmatrix} a_1 a_2 & b \\ 0 & d_1 d_2 \end{pmatrix}$$

$\delta(a_1' b_2 + b_1 d_2')$

and $b = a_1 b_2 + b_1 d_2$

<u>Recall</u> $f|_{\delta A} = \left(\frac{\delta}{|\delta|}\right)^k f|_A$

so we can get rid of δ

Now as b_1 runs over the values mod d_1

b_2 ———————————— δd_2

$b = a_1 b_2 + b_1 d_2$ runs over the values mod $d_1 d_2$, and each value is obtained δ times, so we have

$$mn\, T_m T_n = \sum_{\substack{\delta \mid mn \\ (a_1, d_2) = 1}} \chi(\delta) \delta^{k+1} \sum_{\substack{a_1 d_1 = m/\delta \\ a_2 d_2 = n/\delta}} (a_1 a_2)^k \sum_{b (d_1 d_2)} \begin{pmatrix} a_1 a_2 & b \\ 0 & d_1 d_2 \end{pmatrix}$$

Put $a = a_1 a_2$, $d = d_1 d_2$. $\Rightarrow$ $ad = mn/\delta^2$.

Conversely for each factorisation $ad = \frac{mn}{\delta^2}$, $\exists$ unique factorisations $a = a_1 a_2$, $d = d_1 d_2$ with $(a_1, d_2) = 1$, $a_1 d_1 = m/\delta$, $a_2 d_2 = n/\delta$, Given by $a_1 = \dfrac{m}{(m,\delta d)}$ and $a_2 = \dfrac{\delta d}{(m, \delta d)}$. This gives

$$mn\, T_m T_n = \sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k+1} \underbrace{\sum_{ad = \frac{mn}{\delta^2}} a^k \sum_{b \bmod d} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}_{T_{\frac{mn}{\delta^2}} \cdot \frac{mn}{\delta^2}}$$

$$= \sum_{\delta \mid (m,n)} \chi(\delta)\, \delta^{k-1}\, T_{\frac{mn}{\delta^2}} \qquad \blacksquare$$

<u>Cor</u> $T_m T_n = T_n T_m$ (since the expression for $T_m T_n$ is symmetric in m & n)

<u>Exercise</u> (Fancy Moebius inversion)

$$T_m T_n = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}$$

$$\Rightarrow T_{mn} = \sum_{d \mid (m,n)} \mu(d)\, \chi(d)\, d^{k-1}\, T_{\frac{m}{d}}\, T_{\frac{n}{d}}$$

Then ① $T_{mn} = T_n T_m$ when $(m,n) = 1$ multiplicativity

② $T_{p^{\nu+1}} = T_{p^\nu} T_p - \chi(p)\, p^{k-1}\, T_{p^{\nu-1}}$

$m = p^\nu,\ n = p \Rightarrow (m,n) = p$

③ (Exercise)

$$\sum_{\nu=0}^{\infty} T_{p^\nu} X^\nu = \left(1 - T_p X - \chi(p)\, p^{k-1} X^2\right)^{-1}$$

With $X = p^{-s}$, this gives

$$\sum_{\nu=0}^{\infty} T_{p^\nu}\, p^{-s} = \left(1 - T_p\, p^{-s} - \chi(p)\, p^{k-1} p^{-2s}\right)^{-1} \quad !$$

Hecke operators for $\Gamma = SL_2(\mathbb{Z})$

Here $\displaystyle T_n(f) = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} f|_\rho = n^{\frac{k}{2}-1} \sum_{\rho \in \Gamma \backslash G_n} f|_\rho$

since $f|_{\tau\rho} = f|_\rho$ for $\tau \in \Gamma$ & $f \in M_k(\Gamma)$, so we can use any representative

Thm $T_n : M_k(\Gamma) \longrightarrow M_k(\Gamma)$

$T_n : S_k(\Gamma) \to S_k(\Gamma)$

Proof Let $f \in M_k(\Gamma)$ and $\tau \in \Gamma$. Then

$$(T_n f)|_\tau = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} f|_{\rho\tau} = n^{\frac{k}{2}-1} \sum_{\rho' \in \Delta_n} f|_{\tau'\rho'}$$

and $f|_{\tau'} = f$ since $f \in M_k(\Gamma)$

Then $(T_n f)|_\tau = T_n(f)$.

For cusp forms, we saw that $a_n(0) = \sigma_{k-1}(n)\, a(0)$ ↗ for f

↘ for $T_n(f)$

so if $a(0) = 0$

Then $a_n(0) = 0$ ∎

Thm (Hecke) In the space $S_k(\Gamma)$ of cusp forms for the modular group, $\exists$ an orthonormal basis $\mathcal{F}$ which consists of eigenfunctions for all the Hecke operators T_n.

We will use (without proof, see Iwaniec chapter 2–3)

① $S_k(\Gamma)$ is generated by the Poincaré series

$$P_m(z) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma} j_\tau(z)^{-k} e(m\tau z), \quad m \geq 1$$

What are the representatives for $\Gamma_\infty \backslash \Gamma$?

For any $(c, d) \in \mathbb{Z}^2$ st $(c,d) = 1$, $\exists$ a, b st $ad - bc = 1$

ie $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

a Γ_∞

Also, $\pm\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} * & * \\ c & d \end{pmatrix}$,

and the representatives are $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\}$ $c, d \in \mathbb{Z}^2$, $(c,d) = 1$, (c,d) & not $-(c,d)$

where $\underline{a, d \text{ are fixed}}$ by c & d.

$\underline{m=0}$ $\quad P_0(z) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma} j_\tau(z)^{-k} = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ (c,d) \in \mathbb{Z}^2}} \frac{1}{(cz+d)^k} = E_k(z)$

since $E_k(z) = \dfrac{G_k(z)}{2\zeta(k)}$. Then

$$E_k(z) 2\zeta(k) = \sum_{(c,d)=1} \frac{1}{(cz+d)^k} \sum_{n=1}^{\infty} \frac{1}{n^k}$$

$$= \sum_{n=1}^{\infty} \sum_{(c,d)=n} \frac{1}{(cz+d)^k}$$

$$= \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d) \neq (0,0)}} \frac{1}{(cz+d)^k} = G_k(z)$$

② The Hilbert Space of Cusp forms

Invariant measure on $\mathcal{H}$

Prop Let $U \subseteq \mathcal{H}$ and $f: U \to \mathbb{C}$ be a continuous fct and $\gamma \in GL_2^+(\mathbb{R})$. Then

$$\int_U f(z)\,\frac{dx\,dy}{y^2} = \int_{z \in \gamma^{-1}(U)} f(\gamma z)\,\frac{dx\,dy}{y^2}$$

Proof Compute the Jacobian with the Cauchy Rieman Eqs.

Prop Let $f, g \in M_k(\Gamma)$. Then $F(z) = f(z)\overline{g(z)}\,(\operatorname{Im} z)^k$ is γ invariant, ie $F(\gamma z) = F(z)$.

Furthermore, if $\boxed{f \text{ or } g \in S_k(\Gamma)}$, then $F(z)$ is bounded

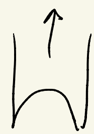
proof

$$\left.\begin{aligned} f(\gamma z) &= (cz+d)^k f(z) \\ \overline{g(\gamma z)} &= \overline{(cz+d)}^k\,\overline{g(z)} \\ \operatorname{Im}(\gamma z)^k &= \frac{\operatorname{Im}(z)^k}{|cz+d|^{2k}} \end{aligned}\right] \Rightarrow F(\gamma z) = F(z)$$

Since Γ is Γ-invariant, it suffices to show it is bounded on $\Gamma \backslash \mathcal{H}$

or that $F(z)$ is bounded as $z \to \infty$,

or $F(q)$ is bounded as $q \to 0$



$$f(q) = \sum_{n=1}^{\infty} a(n)\,e(nz) = e^{2\pi i x} e^{-2\pi y} \sum_{n=0}^{\infty} a(n)\,e((n-1)z)$$

and $|f(z)\overline{g(z)}\,y^k| \ll e^{-2\pi y}\,y^k$ bounded as $y \to \infty$

Def Let $f, g \in S_k(\Gamma)$. The Petersson Inner product is defined by

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2}$$

Inner Product $\langle f, g\rangle = \overline{\langle g, f\rangle}$

$\langle a_1 f_1 + a_2 f_2, g\rangle = a_1\langle f_1, g\rangle + a_2\langle f_2, g\rangle$

$\langle f, f\rangle > 0$ unless $f = 0$

Here $$\langle f, f\rangle = \int_{\Gamma\backslash\mathcal{H}} |f(z)|\, y^k \frac{dx\,dy}{y^2} = \|f\|^2$$

Thm (Iwaniec § 3.3) If $f \in M_k(\Gamma)$, then

$$\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a(m), \quad \text{where } f(z) = \sum a(n)\, e(nz)$$

Cor $$\langle T_n f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a_n(m) = a_m(n)$$ (as we saw)

$$(T_n f)(z) = \sum_{n=0}^{\infty} a_n\, e(nz)$$

Thm For $k \geq 2$, $\boxed{m \geq 0}$ and $n \geq 1$, we have

$$T_n P_m = \sum_{d \mid (m,n)} (n/d)^{k-1} P_{\frac{mn}{d^2}}$$

proof $$(T_n P_m)(z) = n^{\frac{k}{2}-1} \sum_{\tau \in \Gamma_\infty\backslash\Gamma}\ \sum_{\rho \in \Gamma\backslash G_n} j_\tau(z)^{-k} e(m\tau z)\Big|_\rho$$

$$= n^{\frac{k}{2}-1} \underbrace{\sum_{\tau \in \Gamma_\infty\backslash\Gamma}\ \sum_{\rho \in \Gamma\backslash G_n}}_{g \in \Gamma_\infty\backslash G_n} j_{\tau\rho}(z)^{-k} e(m\tau\rho z)$$

Let H be a set of coset reps for $\Gamma_\infty\backslash\Gamma$

G ——————— $\Gamma\backslash G_n$ ie Δ_n

Then HG is a set of coset representatives for $\Gamma_\infty \backslash G_n$, but also GH. (intertwining?).

Then $T_n P_m(z) = n^{k-1} \sum_{\rho \in \Delta_n} \sum_{\tau \in H} j_{\rho\tau}(z)^{-k} e(m\rho\tau z)$

$$= n^{k-1} \sum_{ad=n} d^{-k} \sum_{b \bmod d} \sum_{\tau \in H} j_\tau(z)^{-k} e\left(m \frac{a\tau z + b}{d}\right)$$

since $\rho\tau = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ dc' & dd' \end{pmatrix}$

ie $j_{\rho\tau}(z) = d(c'z + d')$

and again $\sum_{b \bmod d} e\left(\frac{mb}{d}\right) = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$

so we get

$$= n^{k-1} \sum_{\substack{ad=n \\ d \mid m}} d^{1-k} \underbrace{\sum_{\tau \in H} j_\tau(z)^{-k} e\left(\frac{am}{d} \tau z\right)}_{P_{\frac{am}{d}}(z) = P_{\frac{mn}{d^2}}(z)}$$

We can use this for $\boxed{m=0}$. This gives

$$T_n E_k = \sum_{d \mid n} \left(\frac{n}{d}\right)^{k-1} E_k = \sigma_{k-1}(n) E_k$$

ie E_k is an eigenfunction for all the Hecke operators with eigenvalues $\sigma_{k-1}(n)$ [which are the Fourier coefficients]

Cor $m^{k-1} T_n P_m = n^{k-1} T_m P_n$

Similarly, using $\Gamma(k-1)(4\pi m)^{k-1} \langle T_n f, P_m \rangle = a_n(m) \overset{\text{we proved}}{=} a_m(n)$ we get:

Prop $m^{k-1} \langle T_n f, P_m \rangle = n^{k-1} \langle T_m f, P_n \rangle$

Thm The Hecke operators T_n acting on the space of cusp forms for the modular group are self-adjoint ie

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle \quad \forall f, g \in S_k(\Gamma)$$

Proof Since $S_k(\Gamma)$ is spanned by the Poincaré series and properties of the inner product, it suffices to show that for the Poincaré series. By the above,

$$\langle T_n P_m, P_\ell \rangle = \langle T_n P_\ell, P_m \rangle = \overline{\langle P_m, T_n P_\ell \rangle}$$

$$= \langle P_m, T_n P_\ell \rangle$$ since the FC of Poincaré series are real

Thm (from linear algebra)

Let S be a finite dimensional Hilbert space over $\mathbb{C}$ $(S_k(\Gamma))$ and $\mathcal{T}$ be a family of commuting normal operators $T: S \longrightarrow S$ (the Hecke operators T_n).

T commutes with the adjoint T^* and here $T = T^*$

Then $\exists$ an orthonormal basis (w/r to Peterson Inner product) $\mathcal{F}$ of S which consists of common eigenfunctions for all the Hecke operators T_n

ie $\forall f \in S$, $T_n(f) = a_f(n)\, f$

eigenvalues

eigenfunction

Proof. Let $\mathcal{F} = \{f_i\}_{i \le v}$ be an orthonormal basis of S.

Then $Tf_i = \sum_j \lambda_{ij}(T) f_j$ for some $\lambda_{ij}(T) \in \mathbb{C}$.

Then, T corresponds to the matrix $\Lambda(T) = (\lambda_{ij}(T))$ and $\Lambda(T)$ is a <u>normal matrix</u>: it commutes with its adjoint. Furthermore the matrices commute.

<u>Then</u> $\exists$ a unitary matrix $U \in M_v(\mathbb{C})$ <u>st</u> $U^{-1} T U$ is diagonal for every $T \in \mathcal{T}$, <u>ie</u>

$T(f_i) = a_i f_i$ for all $f \in \mathcal{F}$.

<u>Thm</u> (Hecke) In the space $S_k(\Gamma)$ of cusp form for the modular group, $\exists$ an orthonormal basis $\mathcal{F}$ which consists of eigenfunctions for all the Hecke operators.

Those are called <u>Hecke eigenforms</u>

Let $f \in S_k(\Gamma)$ be a Hecke eigenform <u>ie</u>

$$T_n(f) = \lambda_f(n) f \quad \text{for } n = 1, 2, 3, \ldots$$

Suppose that $f(z) = \sum_{m=1}^{\infty} a(m) e(mz)$.

Then $$\lambda_f(n) a(m) = \sum_{d \mid (m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)$$

<u>$m=1$</u> $\quad \lambda(n) a(1) = a(n)$!!!

$\Rightarrow a(1) \neq 0$ if $f \neq 0$

$\underline{\text{Ex}}$ $S_{12}(\Gamma) = \mathbb{C}\,\Delta(z)$

$\Rightarrow$ $\Delta(z)$ is an Hecke eigenform and

$$T_n\,\Delta(z) = \tau(n)\,\Delta(z) \qquad (\text{since } \tau(1)=1)$$

(prove in chapter 3-4 of Iwaniec using Kloosterman sums)